IUPUI
Math Competition
individual Problems

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Question 1

Question 1:

Solution :
(a)

$$\begin{aligned} p(y+h) &= (y+h)^3 + (y+h)^2 + y + h + 1 \\ &= (y^3 + 3y^2h + 3yh^2 + h^3) + (y^2 + 2yh + h^2) + y + h + 1 \\ &= y^3 + y^2(3h+1) + y(3h^2 + 2h + 1) + h^3 + h^2 + h + 1 \qquad (1) \end{aligned}$$

We can see from (1) that $p(y+h)$ has no y^2 term when $3h+1=0$ or when $h = -\frac{1}{3}$

(b)
We can also see from (1) that $p(y+h)$ has no y term when $3h^2 + 2h + 1 = 0$.

But the descriminant, $2^2 - 4(3)(1) = -8$, is negative, so the quadratic has no zeros. This implies that there does not exist an h such that $p(y+h)$ has no y term.

(c)

$$p_n(y+h) = (y+h)^n + (y+h)^{n-1} + ... + y + h + 1$$

Notice that we only need to look at the $(y+h)^n$ and $(y+h)^{n-1}$ terms if we want every term involving y^{n-1}

n=0	1					
n=1	1	1				
n=2	1	2	1			
n=3	1	3	3	1		
n=4	1	4	6	4	1	
n=5	1	5	10	10	5	1

For the $(y+h)^n$ term we know that there will always be an h coefficient on the y^{n-1} term, and we can see from Pascal's triangle that the other coefficient of the y^{n-1} term is simply n (highlighted in red).

And for the $(y+h)^{n-1}$ term the coefficient of the y^{n-1} term is 1 (highlighted in green).

So the y^{n-1} term of $p_n(y+h)$ is

$$y^{n-1}(nh+1)$$

Now we can use the same strategy as in part (a) and find when $nh+1=0$ which yields

$$h = -\frac{1}{n}$$

(d)
Another look at Pascal's triangle shows us that the y term in $p_n(y+h)$ is

$$y(nh^{n-1} + (n-1)h^{n-2} + ... + 2h + 1)$$

If we define $f_n(h) = nh^{n-1} + (n-1)h^{n-2} + ... + 2h + 1 = \sum_{k=0}^{n} kh^{k-1}$ then we want to see for what n there exists an h such that $f_n(h) = 0$

Also notice that

$$\begin{aligned}\lim_{n\to\infty} f_n(h) &= \sum_{k=0}^{\infty} kh^{k-1} \\ &= \sum_{k=0}^{\infty} \frac{d}{dh}(h^k) \\ &= \frac{d}{dh}\sum_{k=0}^{\infty} h^k \\ &= \frac{d}{dh}(\frac{1}{1-h}) \\ &= \frac{1}{(1-h)^2} \text{ for } -1 < h < 1\end{aligned}$$

Case 1: n is even
$\implies$ $n-1$ is odd
$\implies$ $f_n(h)$ is an odd degree polynomial
$\implies$ $f_n(h)$ has at least 1 real root
$\implies$ There exists an h such that $f_n(h) = 0$

Case 2: n is odd
$\implies$ $n-1$ is even
$\implies$ $f_n(h)$ is an even degree polynomial
Since $f_n(h)$ has only positive coefficients, if $f_n(h)$ has any zeros, they must be when $h < 0$. So assume $h < 0$.
We can also see that $f_n(h)$ becomes an alternating series approximating $\frac{1}{(1-h)^2}$ when $h < 0$.
And because $n-1$ is even, the last term of $f_n(h)$, nh^{n-1}, is positive which makes $f_n(h)$ an overestimate of $\frac{1}{(1-h)^2}$.
In other words:

$$f_n(h) \geq \frac{1}{(1-h)^2}$$

And since $\frac{1}{(1-h)^2} > 0$, $f_n(h) > 0$. So $f_n(h)$ has no zeros.

Result: It is only possible to find an h such that $p_n(y+h)$ has no y term when n is even.

Question 3

Question 3:

Solution :

Looking at a table of some values for $f(n)$ can be useful while solving the problem.

n	0	1	2	3	4	5	6	7	8
$f(n)$	1	0	1	0	0	0	1	0	1

Given a $k \in \mathbb{Z}^+$, let $B(k)$ denote the bitstring of the concatenation of values of $f(n)$ from $n = 0$ to $n = k - 1$.

Example: $B(5) = 10100$ because 1, 0, 1, 0, and 0 are the first 5 values of $f(n)$

We can see some patterns when looking specifically at bit strings of length 3^n.

n	$B(3^n)$
0	1
1	101
2	101000101
3	101000101000000000101000101

From the table of bitstrings, we can hypothesize that the next 3^k values of $f(n)$ following $f(3^k - 1)$ are all 0s. In more mathematical terms, we want to prove that: $3^k \leq n < 2(3^k) \implies f(n) = 0$ where $n, k \in \mathbb{Z}^+$

Proof by induction.
Base case: $k = 1$. $f(3) = 0$, $f(4) = 0$, and $f(5) = 0$ so base case holds.
Assume true for $m = k$.
$m + 1$ case:
$3^{m+1} \leq n < 2(3^{m+1})$

Case 1: $n \equiv 0 \mod 3$
$f(n) = f(3(3^m + a))$ where $0 \leq a < 3^m$
$= f(3^m + a) = 0$ by hypothesis.
Case 2: $n \equiv 1 \mod 3$
By definition, $f(3n + 1) = 0$ so $f(n) = 0$.
Case 3: $n \equiv 2 \mod 3$
By definition, $f(3n + 2) = f(n) = f(3n)$ which is equivalent to case 1. □

A corollary of this is that $f(3^n) = 0$.

By looking at the symmetry of the bitstring table, we can see that

$$\sum_{n=0}^{3^k-1} f(n) = 2 \sum_{n=0}^{3^{k-1}-1} f(n) = 2^k$$

And since in general $f(3^n) = 0$, we can clean it up a little to be

$$\sum_{n=0}^{3^k} f(n) = 2\sum_{n=0}^{3^{k-1}} f(n) = 2^k$$

With both of these important facts, we can start to solve the problem.

Observe the fact that

$$\sum_{n=0}^{3^{48}} f(n) = 2^{48}$$

Note that 3^{48} is the largest power of 3 less than 10^{23}.
If we take into account the fact that there are 3^{48} zeros right after $f(3^{48}-1)$ and that $10^{23} \leq 2(3^{48})$, then we know $B(2(3^{48}))$ looks like this:

$$\overbrace{\underbrace{\overbrace{101000...000101}^{\text{first } 3^{48} \text{ numbers}}0000..}_{\text{first } 10^{23} \text{ numbers}}.0000}^{\text{first } 2(3^{48}) \text{ numbers}}$$

So we can say:

$$\begin{aligned} 10^{23} \leq 2(3^{48}) &\implies \sum_{n=0}^{3^{48}} f(n) = \sum_{n=0}^{10^{23}} f(n) \\ &\implies \sum_{n=0}^{10^{23}} f(n) = \underline{2^{48}}. \end{aligned}$$

Question 4

Question 4:

Solution :
(a)
To solve this problem, I analyzed arrays of smaller dimensions and then observed patterns that gave insight on arrays of higher dimensions.

If we number each cube in a $3 \times 3 \times 3$ array with the step in which its labeled, we get this:

3	4	5	4	5	6	5	6	7
2	3	4	3	4	5	4	5	6
1	2	3	2	3	4	3	4	5

We can see that it takes 7 steps to label a $3 \times 3 \times 3$ cube because there are 7 distinct numbers in the diagram.

3	4	5	4	5	6	5	6	7
2	3	4	3	4	5	4	5	6
1	2	3	2	3	4	3	4	5

If we color code the diagram, we can also see a general pattern, which is that the amount of distinct numbers in an $n \times n \times n$ diagram is

$$(n) + (n-1) + (n-1)$$

or $3n - 2$.

So for a $10 \times 10 \times 10$ array, it takes $3(10) - 2$ or <u>28</u> steps to label every cube

(b)
To solve this problem, I used a computer to simulate the steps.

First, I noticed that it is really only necessary to add the cubes directly below, south, and west of the current cube, as all the other surrounding cubes will always be 0.

Then, I wrote the following Python3 code to compute an entire $10 \times 10 \times 10$ array of cubes and output the top-north-east label:

```
from itertools import product
n=10
a = [[[0 for i in range(n)] for j in range(n)] for k in range(n)]
a[0][0][0] = 1
for x,y,z in product(range(n),repeat=3):
        a[x][y][z] += a[x-1][y][z] + a[x][y-1][z] + a[x][y][z-1]
print(a[n-1][n-1][n-1])
```

Note that in the code, the order in which the cubes are iterated is different from the order described in the problem. The code still works because of the fact that when a specific cube is calculated, the cubes directly below, south, and west of it have already been calculated.

The output of the program is 227873431500. Which is the answer to the problem.

The code could easily be altered to compute any $n \times n \times n$ array by simply changing the second line.

Question 5

Question 5:

A common topic that is discussed when first learning about fractals is that fractals are structures that can be seen all throughout nature. The examples usually listed are the patterns of tree branches, snowflakes, flowers, or even the geographical coastlines of land caused by erosion. Mathematician Benoit Mandelbrot took a particular interest in the last example. Mandelbrot was interested in all sorts of mathematics, but he is best known for his work in fractal geometry. In his 1967 paper "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension", Mandelbrot introduced the idea of a fractal "dimension", which extended the conventional way of thinking about dimensions and their constraint to the natural numbers. A line would have a dimension of 1, and a square would have a dimension of 2, but a curve would have a dimension value between 1 and 2 based on its self-similarity. A simple way to calculate the dimension of a fractal is to do what is called the Box Counting method. The Box Counting method is mostly used when calculating the fractal dimension of shapes found in nature, because the dimension of mathematically pure fractals can be calculated in more precise ways. In the Box Counting method, a grid of boxes is overlaid onto the fractal (the coastline) and the number of boxes that cover the fractal is counted. If we let N be the number of boxes that cover any portion of the coastline and M be the multiplicative inverse of the size of each box, then the fractal dimension will be approximately $log(N)/log(M)$. Making the size of the boxes smaller will increase the accuracy of the approximation. Using this method, we can calculate the fractal dimension of the west coast of Britain to be about 1.25, as described in Mandelbrot's paper. The idea of calculating fractal dimensions has proven to be useful in mathematics. Fractal dimensions give us a quantitative method to measure "roughness" in nature. Smoother curves will have fractal dimensions closer to 1, while rougher curves will have fractal dimensions closer to 2.

The value of fractal dimension extends beyond analyzing objects found in nature: enter Jackson Pollock, the American painter and pioneer of the abstract expressionist movement during the 1940-50s. Pollock utilized a painting technique where he let paint drip from a stick, brush, or even directly from the paint can, rather than making direct contact between the paint brush and the canvas. Today, original Jackson Pollock pieces can be valued up to $500 million. As a result, there have been several instances where missing pieces from Pollock's collection have been "found", only to be deemed forgeries. When "found" Pollock pieces have surfaced in the past, art experts specializing in Jackson Pollock were necessary to determine the validity of the paintings. However, in 1989, several professors from the University of Oregon showed how fractal dimensions can be used to objectively determine if pieces are authentic Pollock paintings or forgeries. In a paper titled "Authenticating Pollock paintings using fractal geometry", the team of professors detailed how the theory of fractal dimension found in Mandelbrot's original paper can be implemented in this ingenuitive way. In the paper, the authors use the statistical self-similarity of Pollock's work to analyze the fractal dimension in the different layers and compare that to unauthenticated paintings. They used the Box Counting strategy to calculate the fractal dimension and found that each layer of Pollock's paintings consists of two distinct dimension values based on the scale of the boxes: D_L and D_D. By taking into account the values of D_L and D_D, the deviation of the values, and the scale at which they change, the authors had a 100% success rate in identifying already accepted Pollock paintings. Additionally, the authors were able to analyze 14 unauthenticated paintings and conclude that none of them satisfied the characteristics they previously quantified in authenticated Pollock paintings.

More recent studies in 2015 have done more extensive testing with a 93% accuracy rate. This impressive feat goes to show that seemingly pure mathematical concepts can have incredible realworld applications.

Sources:

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